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On an n-Dimensional Quadratic Spline Approximation

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In this paper a multi-variate spline interpolational method on a rectangular grid is presented. The method is based on the use of a special continuous piecewise polynomial which is quadratic in each variable. In addition to approximation properties, the shape preserving characteristics and stability of the method have been proved.

INTRODUCTION

The objective of this paper is to construct a spline approximational method for functions in several variables. Various spline approximational methods have been worked out by several authors. For references addressing this problem see the monumental bibliography of [4]. For functions of several variables the most useful techniques on a rectangular grid are the tensor product methods and the blending methods. Here we present a different method. Our method produces an interpolant on a rectangular grid in n dimensions, which is a special piecewise quadratic polynomial in each variable, and it is continuous. By using a recursive formula one gets a particularly simple expression for the approximant, including only the nodal function values (and maybe the nodal derivative values), which is very useful for practical reasons and for applying computing procedures. We prove approximation theorems for the spline function, which show that the order of the approximation is the best possible, depending on the smoothness of the function, and even the constants in the estimates can be calculated easily. Another feature of this method is that the approximant possesses some shape-preserving properties. Finally, the stability of the construction makes it possible to apply it to numerical solution of partial differential equations, which is also illustrated with examples.

We note that similar techniques can be found in [11] in the two-dimensional case.

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1. Notations

In what follows, $\mathbb{R}$, $\mathbb{Z}$, and $\mathbb{N}$ denote the set of reals, the set of the integers, and the set of natural numbers (including zero), respectively. For any vector $\mathbf{x}$ in R^n we denote its jth component by $(\mathbf{x})_j = x_j$, that is, $\mathbf{x} = (x_1, x_2, ..., x_n)$. Addition, multiplication, and inequality between vectors will be defined componentwise. For $\mathbf{a}$, $\mathbf{b}$ in $\mathbb{R}^n$ we write $[\mathbf{a}, \mathbf{b}] = \{\mathbf{x} \in \mathbb{R}^n: \mathbf{a} \leqslant \mathbf{x} \leqslant \mathbf{b}\}$, and $\mathbf{a}^{\mathbf{b}} = \prod_{j=1}^n (a_j)^{b_j}$, where $0^0 = 1$. The zero vector will be denoted by $\mathbf{0}$. We let $\mathbf{e} = (1, 1, ..., 1)$, further $\mathbf{e}_j$ denotes the vector, whose jth coordinate equals 1, the others being zero ($j = 1, 2, ..., n$). If u: $\mathbb{R}^n \to \mathbb{R}$, $\mathbf{h}$, $\mathbf{k}$ are arbitrary elements of $\mathbb{R}^n$, then let $\Delta_{\mathbf{h}}^{\mathbf{k}}$ denote the difference operator

$$\Delta_{\mathbf{h}}^{\mathbf{k}} u(\mathbf{t}) = \Delta_{h_1, ..., h_n}^{k_1, ..., k_n} u(t_1, ..., t_n) \qquad (t \in \mathbb{R}^n),$$

where $\Delta_{h_1, ..., h_n}^{k_1, ..., k_n}$ is the product of the k_jth iterates of the difference operators with increment h_j in the jth variable, respectively. The modulus of continuity of a given function u is defined as usual, and we denote it by $\omega_d(u)$. Here d stands for the (euclidean) diameter of the set, on which the oscillation of u is considered.

2. Construction of the Spline Function

Let $\{\mathbf{t_i}\}_{\mathbf{i} \in \mathbb{Z}^n}$ be an equidistant subdivision of $\mathbb{R}^n$ with $\mathbf{h} = (h_1, ..., h_n)$, that is, $(\mathbf{t}_{\mathbf{i}+\mathbf{e}_j} - \mathbf{t_i})_j = h_j$. Let $\{u_{\mathbf{i}}\}_{\mathbf{i} \in \mathbb{Z}^n}$ and $\{u_{\mathbf{i}}^{(\mathbf{e}_j)}\}_{\mathbf{i} \in \mathbb{Z}^n}$ $(j = 1, ..., n)$ be given systems of real numbers. We put $d = [\sum_{j=1}^n h_j^2]^{1/2}$, the diameter corresponding to this subdivision.

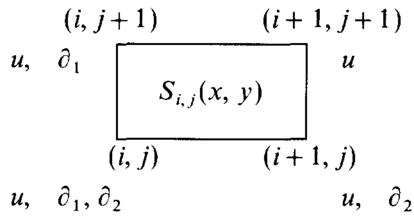
For all $\mathbf{t}$ in $[\mathbf{t_i}, \mathbf{t}_{\mathbf{i}+\mathbf{e}}]$ let

$$S_{\mathbf{i}}(\mathbf{t}) = \sum_{\mathbf{k} \in \mathbb{K}} A_{\mathbf{i}}^{(\mathbf{k})} (\mathbf{t} - \mathbf{t_i})^{\mathbf{k}}, \tag{1$_n$}$$

where $\mathbb{K}$ is the set of all n-dimensional multi-indices $\mathbf{k}$, with $0 \leqslant \mathbf{k} \leqslant 2\mathbf{e}$ and $k_j = 2$ for at most one j; that is, $S_{\mathbf{i}}$ is a special quadratic polynomial in each variable. Further, the coefficients $A_{\mathbf{i}}^{(\mathbf{k})}$ are to be chosen satisfying the conditions

$$\begin{aligned} S_{\mathbf{i}}(\mathbf{t}_{\mathbf{i}+\mathbf{l}}) &= u_{\mathbf{i}+\mathbf{l}} && \text{for} \quad \mathbf{0} \leqslant \mathbf{l} \leqslant \mathbf{e}, \\ \partial_j S_{\mathbf{i}}(\mathbf{t}_{\mathbf{i}+\mathbf{l}}) &= u_{\mathbf{i}+\mathbf{l}}^{(\mathbf{e}_j)} && \text{for} \quad \mathbf{0} \leqslant \mathbf{l} \leqslant \mathbf{e}, \quad (\mathbf{l})_j = 0. \end{aligned} \tag{2$_n$}$$

The numbers of the unknown coefficients $A_{\mathbf{i}}^{(\mathbf{k})}$ and of the conditions (equations) are equal to $2^n + n2^{n-1}$. In the two-dimensional case the conditions can be illustrated in



that is, we have the equations

$$S_{i,j}(x_p, y_q) = u_{p,q},$$

$$\partial_1 S_{i,j}(x_i, y_q) = u_{i,q}^{(1,0)},$$

$$\partial_2 S_{i,j}(x_p, y_j) = u_{p,j}^{(0,1)},$$

for $p = i,\ i+1$, and $q = j,\ j+1$; further the unknown coefficients are $A_{i,j}^{(0,0)}$, $A_{i,j}^{(0,1)}$, $A_{i,j}^{(1,0)}$, $A_{i,j}^{(2,0)}$, $A_{i,j}^{(1,1)}$, $A_{i,j}^{(0,2)}$, $A_{i,j}^{(2,1)}$, $A_{i,j}^{(1,2)}$.

LEMMA 1. *There exist unique constants $A_{\mathbf{i}}^{(\mathbf{k})}$ such that the functions $S_{\mathbf{i}}$ of the form $(1)_n$ satisfy $(2)_n$.*

Proof. We show by induction, with respect to dimension n, that the coefficients $A_{\mathbf{i}}^{(\mathbf{k})}$ are uniquely determined by condition $(2)_n$. For $n = 1$ this is trivial. In the $(n+1)$-dimensional case it follows by induction that all those coefficients $A_{\mathbf{i}}^{(\mathbf{k})}$ are uniquely determined, where $\mathbf{k}$ has at least one zero coordinate. For the remaining coefficients $A_{\mathbf{i}}^{(\mathbf{k})}$ we have a system of linear equations with the regular matrix

$$\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 0 & \cdots & 1 & 1 \\ & & & \cdots & & \\ 1 & 1 & 1 & \cdots & 1 & 0 \end{pmatrix};$$

hence, existence and uniqueness are proved. ∎

Now we define an n-dimensional quadratic spline function S (corresponding to the knots $\{\mathbf{t}_{\mathbf{i}}\}$ and to the systems $\{u_{\mathbf{i}}\}$, $\{u_{\mathbf{i}}^{(\mathbf{e}_j)}\}$) on $\mathbb{R}^n$ as follows: for all $\mathbf{t}$ in $[\mathbf{t}_{\mathbf{i}}, \mathbf{t}_{\mathbf{i}+\mathbf{e}})$ let

$$S(\mathbf{t}) = S_{\mathbf{i}}(\mathbf{t}). \qquad (3)_n$$

Remark 1. For instance, the two-dimensional quadratic spline function of the form $(1)_2$–$(3)_2$ can be expressed as follows: for any $x \in [x_i, x_{i+1}]$ and $y \in [y_j, y_{j+1}]$ we have

$$\begin{aligned} S(x, y) = S_{i,j}(x, y) = {} & (1-s)\{u_{i,j} + t[hu_{i,j}^{(1,0)} + t(u_{i+1,j} - u_{i,j} - hu_{i,j}^{(1,0)})]\} \\ & + s\{u_{i,j+1} + t[hu_{i,j+1}^{(1,0)} + t(u_{i+1,j+1} - u_{i,j+1} - hu_{i,j+1}^{(1,0)})]\} \\ & + (1-s)s\{(lu_{i,j}^{(0,1)} - u_{i,j+1} + u_{i,j})(1-t) \\ & + (lu_{i+1,j}^{(0,1)} - u_{i+1,j+1} + u_{i+1,j})t\}, \end{aligned}$$

where we have used the notation $x_{i+1} - x_i = h$, $y_{j+1} - y_j = l$, $t = (x - x_i)/h$, and $s = (y - y_j)/l$.

More generally, it is easy to see (again by the uniqueness part of Lemma 1), that the $n+1$-dimensional spline function $S^{(n+1)}$ can be expressed with the help of the n-dimensional $S^{(n)}$ as

$$\begin{aligned} & S_{\mathbf{i}}^{(n+1)}(t_1, ..., t_n, t_{n+1}) \\ & \quad = v_{n+1} S_{\mathbf{i}+\mathbf{e}_{n+1}}^{(n)}(t_1, ..., t_n) + (1 - v_{n+1}) S_{\mathbf{i}}^{(n)}(t_1, ..., t_n) \\ & \qquad + (v_{n+1} - 1) v_{n+1} \sum_{\substack{\mathbf{0} \leqslant \mathbf{l} \leqslant \mathbf{e} \\ l_{n+1} = 0}} \prod_{j=1}^{n} w_j [u_{\mathbf{i}+\mathbf{l}+\mathbf{e}_{n+1}} - u_{\mathbf{i}+\mathbf{l}} - h_{n+1} u_{\mathbf{i}+\mathbf{l}}^{(\mathbf{e}_{n+1})}], \end{aligned}$$

where

$$v_j = \frac{(\mathbf{t})_j - (\mathbf{t}_{\mathbf{i}})_j}{h_j}, \qquad w_j = \begin{cases} v_j, & \text{if} \quad l_j = 1 \\ 1 - v_j, & \text{if} \quad l_j = 0 \end{cases}$$

for $j = 1, ..., n$. This recursive formula will be very useful in proving approximation properties of the spline function, as we see in the next section.

LEMMA 2. *The n-dimensional quadratic spline function S defined by* $(1)_n$–$(3)_n$ *is continuous.*

Proof. It is enough to prove that $S_{\mathbf{i}}(\mathbf{t}) = S_{\mathbf{i}+\mathbf{e}_j}(\mathbf{t})$ holds for all those $\mathbf{t}$ in the interval $[\mathbf{t_i}, \mathbf{t}_{\mathbf{i}+\mathbf{e}}]$ which are not contained in $[\mathbf{t_i}, \mathbf{t}_{\mathbf{i}+\mathbf{e}})$. In other words, we must show that

$$S_{\mathbf{i}}(\mathbf{t}) = S_{\mathbf{i}+\mathbf{e}_j}(\mathbf{t})$$

holds for all $\mathbf{t}$ with the property $(\mathbf{t})_j = (\mathbf{t_i})_j + h_j$ $(j = 1, ..., n+1)$. We restrict ourselves to the case $j = n+1$. We write $\mathbf{t} = (\mathbf{t}', (\mathbf{t_i})_{n+1})$ with $\mathbf{t}' \in \mathbb{R}^n$. One sees immediately that both functions

$$\mathbf{t}' \rightarrow S_{\mathbf{i}}(\mathbf{t})$$

and

$$\mathbf{t}' \to S_{\mathbf{i}+\mathbf{e}_j}(\mathbf{t})$$

are of the form $(1)_n$ and satisfy the conditions $(2)_n$; hence by the uniqueness part of Lemma 1 they coincide for all $\mathbf{t}'$. ▮

3. Approximation Properties of the Spline Function

In what follows we apply the above spline construction for different choices of the given values $\{u_{\mathbf{i}}\}_{\mathbf{i}\in\mathbb{Z}^n}$ and $\{u_{\mathbf{i}}^{(\mathbf{e}_j)}\}_{\mathbf{i}\in\mathbb{Z}^n}$ $(j=1, ..., n)$. However, the values $\{u_{\mathbf{i}}\}_{\mathbf{i}\in\mathbb{Z}^n}$ will always be the function values of a function $u\colon \mathbb{R}^n \to \mathbb{R}$ at the knots of the subdivision $\{\mathbf{t}_{\mathbf{i}}\}_{\mathbf{i}\in\mathbb{Z}^n}$, that is

$$u_{\mathbf{i}} = u(\mathbf{t}_{\mathbf{i}}) \qquad (4)_n$$

for all $\mathbf{i}$. On the other hand, the values $\{u_{\mathbf{i}}^{(\mathbf{e}_j)}\}_{\mathbf{i}\in\mathbb{Z}^n}$ can be defined quite arbitrarily in several different ways.

Let $u\colon \mathbb{R}^n \to \mathbb{R}$ be a function having first order partial derivatives with respect to each variable. We define for all $\mathbf{i}$ in $\mathbb{Z}^n$ and $j = 1, 2, ..., n$

$$u_{\mathbf{i}}^{(\mathbf{e}_j)} = \partial_j u(\mathbf{t}_{\mathbf{i}}). \qquad (5)_n$$

First we study the approximating properties of the respective spline function, depending on the smoothness of u.

Theorem 1. *Let $u\colon \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable. Then the spline function S defined by the conditions $(1)_n$–$(5)_n$ satisfies*

$$|u(\mathbf{t}) - S(\mathbf{t})| \leqslant \tfrac{1}{2} \sum_{j=1}^{n} h_j \omega_d(\partial_j u),$$

and

$$|\partial_k u(\mathbf{t}) - \partial_k S(\mathbf{t})| \leqslant \left(n - \frac{1}{2}\right) \omega_d(\partial_k u) + \frac{1}{2h_k} \sum_{j=1}^{n} h_j \omega_d(\partial_j u)$$

for all $\mathbf{t}$ in $\mathbb{R}^n$ and for $k = 1, ..., n$, where d is the diameter corresponding to the subdivision.

Proof. We prove the theorem by induction with respect to the dimension and for the case $n=1$ we use the following formulas from [6]: if $t \in [t_i, t_{i+1}]$, then

$$S(t)-u(t)=S_i(t)-u(t)=u_i(1-v^2)+u_{i+1}v^2+hu_i'v(1-v)-u(t),$$

$$|S(t)-u(t)| \leqslant \frac{2}{3\sqrt{3}}\, h\omega_h(u') \leqslant \frac{1}{2}\, h\omega_h(u'),$$

and

$$|S_i'(t)-u'(t)| = |2v[(u_{i+1}-u_i)/h-u'(t)]+(1-2v)[u_i'-u'(t)]| \leqslant \omega_h(u'),$$

where $v=(t-t_i)/h$. Hence our statement for $n=1$ follows immediately, and now we consider the statement in $\mathbb{R}^{n+1}$. Let $\mathbf{t} \in [\mathbf{t_i}, \mathbf{t_{i+e}}]$; then we have by induction and by the Lagrange theorem

$$\begin{aligned}
&|u(\mathbf{t})-S(\mathbf{t})| \\
&\quad = |u(t_1, ..., t_n, t_{n+1}) - S_{\mathbf{i}}^{(n+1)}(t_1, ..., t_n, t_{n+1})| \\
&\quad \leqslant v_{n+1}|u(t_1, ..., t_n, (\mathbf{t_{i+e}})_{n+1}) - S^{(n)}_{\mathbf{i}+\mathbf{e}_{n+1}}(t_1, ..., t_n)| \\
&\qquad + (1-v_{n+1})\,|u(t_1, ..., t_n, (\mathbf{t_i})_{n+1}) - S_{\mathbf{i}}^{(n)}(t_1, ..., t_n)| \\
&\qquad + |v_{n+1}[u(t_1, ..., t_n, t_{n+1}) - u(t_1, ..., t_n, (\mathbf{t_{i+e}})_{n+1})] \\
&\qquad + (1-v_{n+1})[u(t_1, ..., t_n, t_{n+1}) - u(t_1, ..., t_n, (\mathbf{t_i})_{n+1})]| \\
&\qquad + (1-v_{n+1})\, v_{n+1} \sum_{\substack{\mathbf{0} \leqslant \mathbf{l} \leqslant \mathbf{e} \\ l_{n+1}=0}} \prod_{j=1}^{n} w_j |u_{\mathbf{i}+\mathbf{l}+\mathbf{e}_{n+1}} - u_{\mathbf{i}+\mathbf{l}} - h_{n+1} u_{\mathbf{i}+\mathbf{l}}^{(\mathbf{e}_{n+1})}| \\
&\quad \leqslant \tfrac{1}{2} \sum_{j=1}^{n} h_j \omega_d(\partial_j u) + (1-v_{n+1})\, v_{n+1} h_{n+1} |\partial_{n+1} u(t_1, ..., t_n, \xi_{n+1}) \\
&\qquad - \partial_{n+1} u(t_1, ..., t_n, \vartheta_{n+1})| \\
&\qquad + (1-v_{n+1})\, v_{n+1} h_{n+1} \omega_d(\partial_{n+1} u) \sum_{\substack{\mathbf{0} \leqslant \mathbf{l} \leqslant \mathbf{e} \\ l_{n+1}=0}} \prod_{j=1}^{n} w_j \leqslant \tfrac{1}{2} \sum_{j=1}^{n+1} h_j \omega_d(\partial_j u),
\end{aligned}$$

where $(\mathbf{t_i})_{n+1} < \xi_{n+1}, \vartheta_{n+1} < (\mathbf{t_{i+e}})_{n+1}$. Here we have used the obvious identity

$$\sum_{\mathbf{0} \leqslant \mathbf{l} \leqslant \mathbf{e}} \prod_{j=1}^{n} w_j = 1.$$

The respective statement for the derivatives can be obtained similarly, and here we prove it for $k=1$. Let $\mathbf{t} \in [\mathbf{t_i}, \mathbf{t_{i+e}}]$; then we have by induction and by the definition of the modulus of continuity

$$
\begin{aligned}
&|\partial_1 u(\mathbf{t}) - \partial_1 S(\mathbf{t})| \\
&\quad = |\partial_1 u(t_1, ..., t_n, t_{n+1}) - \partial_1 S_{\mathbf{i}}^{(n+1)}(t_1, ..., t_n, t_{n+1})| \\
&\quad \leqslant v_{n+1} |\partial_1 u(t_1, ..., t_n, (\mathbf{t_{i+e}})_{n+1}) - \partial_1 S_{\mathbf{i}+\mathbf{e}_{n+1}}^{(n)}(t_1, ..., t_n)| \\
&\qquad + (1 - v_{n+1}) |\partial_1 u(t_1, ..., t_n, (\mathbf{t_i})_{n+1}) - \partial_1 S_{\mathbf{i}}^{(n)}(t_1, ..., t_n)| \\
&\qquad + v_{n+1} |\partial_1 u(t_1, ..., t_n, t_{n+1}) - \partial_1 u(t_1, ..., t_n, (\mathbf{t_{i+e}})_{n+1}| \\
&\qquad + (1 - v_{n+1}) |\partial_1 u(t_1, ..., t_n, t_{n+1}) - \partial_1 u(t_1, ..., t_n, (\mathbf{t_i})_{n+1})| \\
&\qquad + (1 - v_{n+1}) v_{n+1} \sum_{\substack{\mathbf{0} \leqslant \mathbf{l} \leqslant \mathbf{e} \\ l_{n+1} = 0 \\ l_1 = 0}} \prod_{j=2}^{n} w_j \frac{h_{n+1}}{h_1} 2\omega_d(\partial_{n+1} u) \\
&\quad \leqslant \left(n - \frac{1}{2}\right) \omega_d(\partial_1 u) + \frac{1}{2h_1} \sum_{j=1}^{n} h_j \omega_d(\partial_j u) \\
&\qquad + \omega_d(\partial_1 u) + \frac{h_{n+1}}{2h_1} \omega_d(\partial_{n+1} u) \\
&\quad = \left(n + \frac{1}{2}\right) \omega_d(\partial_1 u) + \frac{1}{2h_1} \sum_{j=1}^{n+1} h_j \omega_d(\partial_j u);
\end{aligned}
$$

hence our theorem is proved. ∎

THEOREM 2. *Let $u \colon \mathbb{R}^n \to \mathbb{R}$ be twice continuously differentiable. Then the spline function S defined by the conditions $(1)_n$–$(5)_n$ satisfies*

$$
|u(\mathbf{t}) - S(\mathbf{t})| \leqslant \frac{1}{4} \sum_{j=1}^{n} h_j^2 \omega_d(\partial_j^2 u),
$$

$$
|\partial_k u(\mathbf{t}) - \partial_k S(\mathbf{t})| \leqslant \frac{1}{4} \sum_{j=1}^{n} h_j \omega_d(\partial_k \partial_j u) + \frac{1}{8} \sum_{j=1}^{n} \frac{h_j^2}{h_k} \omega_d(\partial_j^2 u),
$$

and

$$
|\partial_k^2 u(t) - \partial_k^2 S(\mathbf{t})| \leqslant n\omega_d(\partial_k^2 u)
$$

for all $\mathbf{t}$ in $\mathbb{R}^n$ and for $k = 1, ..., n$, where d is the diameter corresponding to the subdivision.

Proof. First we deal with the one-dimensional case. We use the following formula of [6] again: for $t \in [t_i, t_{i+1}]$ we have

$$\begin{aligned} &S(t)-u(t) \\ &\quad = S_i(t)-u(t)=u_i(1-v^2)+u_{i+1}v^2+hu_i'v(1-v)-u(t) \\ &\quad = h^2\left[\int_0^v \psi_1(v,\tau)\,u''(t_i+\tau h)\,d\tau+\int_v^1 \psi_2(v,\tau)\,u''(t_i+\tau h)\,d\tau\right], \end{aligned}$$

where $v=(t-t_i)/h$, $\psi_1(v,\tau)=(1-v)[(1+v)\tau-v]$, $\psi_2(v,\tau)=v^2(1-\tau)$; further

$$|S(t)-u(t)| \leqslant 0{,}046h^2\omega_h(u''),$$

which implies the first statement for $n=1$. In order to estimate the derivatives, we compute

$$\begin{aligned} R'(t)=S_i'(t)-u_i'(t)&=hu''(t)[\psi_1(v,v)-\psi_2(v,v)] \\ &+h\left[\int_0^v\left(\frac{\partial}{\partial v}\psi_1(v,\ \tau)\right)u''(t_i+\tau h)\,d\tau\right. \\ &\left.+\int_v^1\left(\frac{\partial}{\partial v}\psi_2(v,\tau)\right)u''(t_i+\tau h)\,d\tau\right] \\ &=h\left[\int_0^v (2v(1-\tau)-1)\,u''(t_i+\tau h)\,d\tau\right. \\ &\left.+\int_v^1 2v(1-\tau)\,u''(t_i+vh)\,d\tau\right]. \end{aligned}$$

In the first integral the function $v \to 2v(1-\tau)-1$ keeps its sign for $0 \leqslant v < 1/2$, and for $1/2 \leqslant v \leqslant 1$ it changes the sign only for $\tau=\tau^*=(2v-1)/2v$, and in the second integral the function $v \to 2v(1-\tau)$ is non-negative. In order to apply the mean value theorem, we consider the cases $0 \leqslant v < 1/2$ and $1/2 \leqslant v \leqslant 1$ separately. For $0 \leqslant v < 1/2$ we have

$$\begin{aligned} R'(t)&=h\left[u''(\eta_1)\int_0^v (2v(1-\tau)-1)\,d\tau+u''(\eta_2)\int_v^1 2v(1-\tau)\,d\tau\right] \\ &=h[u''(\eta_1)-u''(\eta_2)]\,v(2v-1-v^2), \end{aligned}$$

where $\eta_1, \eta_2 \in (t_i, t_{i+1})$ and for $1/2 \leqslant v \leqslant 1$ it follows that

$$R'(t) = h\left[u''(\xi_1)\int_0^{\tau^*}(2v(1-\tau)-1)\,d\tau\right.$$

$$\left.+u''(\xi_2)\int_{\tau^*}^{v}(2v(1-\tau)-1)\,d\tau + u''(\eta)\int_v^1 2v(1-\tau)\,d\tau\right]$$

$$= h\left[(u''(\xi_1)-u''(\xi_2))\frac{(2v-1)^2}{4v} + (u''(\eta)-u''(\xi_2))\,v(v-1)^2\right],$$

where $\xi_1, \xi_2, \eta \in (t_i, t_{i+1})$, and hence we have in both cases for all t

$$|R'(t)| \leqslant \tfrac{3}{8}h\omega_h(u'').$$

For the estimation of $|R''(t)|$ we have by differentiation

$$R''(t) = \left[\int_0^v 2(1-\tau)u''(t_i+\tau h)\,d\tau + \int_v^1 2(1-\tau)u''(t_i+\tau h)\,d\tau\right] - u''(t)$$

$$= \int_0^1 2(1-\tau)\,u''(t_i+\tau h)\,d\tau - u''(t) = u''(\xi) - u''(t),$$

where $\xi \in (t_i, t_{i+1})$, that is,

$$|R''(t)| = \omega_h(u''),$$

and hence the statements of the theorem are proved for $n=1$. Now we apply induction again with respect to the dimension. Let $\mathbf{t} \in [\mathbf{t}_\mathbf{i}, \mathbf{t}_{\mathbf{i}+\mathbf{e}}]$; then we have by induction and by the second order Taylor formula

$$|u(\mathbf{t}) - S(\mathbf{t})|$$

$$= |u(t_1, ..., t_n, t_{n+1}) - S_\mathbf{i}^{(n+1)}(t_1, ..., t_n, t_{n+1})|$$

$$\leqslant v_{n+1}\,|u(t_1, ..., t_n, (\mathbf{t}_{\mathbf{i}+\mathbf{e}})_{n+1}) - S^{(n)}_{\mathbf{i}+\mathbf{e}_{n+1}}(t_1, ..., t_n)|$$

$$+ (1-v_{n+1})\,|u(t_1, ..., t_n, (\mathbf{t}_\mathbf{i})_{n+1}) - S_\mathbf{i}^{(n)}(t_1, ..., t_n)|$$

$$+ \Big|v_{n+1}[u(t_1, ..., t_n, t_{n+1}) - u(t_1, ..., t_n, (\mathbf{t}_{\mathbf{i}+\mathbf{e}})_{n+1})]$$

$$+ (1-v_{n+1})[u(t_1, ..., t_n, t_{n+1}) - u(t_1, ..., t_n, (\mathbf{t}_\mathbf{i})_{n+1})]$$

$$+ (1-v_{n+1})\,v_{n+1}\sum_{\substack{\mathbf{0}\leqslant\mathbf{1}\leqslant\mathbf{e}\\ l_{n+1}=0}}\prod_{j=1}^{n} w_j(u_{\mathbf{i}+\mathbf{1}+\mathbf{e}_{n+1}} - u_{\mathbf{i}+\mathbf{1}} - h_{n+1}u_{\mathbf{i}+\mathbf{1}}^{(\mathbf{e}_{n+1})})\Big|$$

$$\leqslant \tfrac{1}{4} \sum_{j=1}^{n} h_j^2 \omega_d(\partial_j^2 u) + (1 - v_{n+1}) v_{n+1} h_{n+1} \Big| \partial_{n+1} u(t_1, ..., t_n, (\mathbf{t_i})_{n+1})$$

$$+ \tfrac{1}{2} h_{n+1} v_{n+1} \partial_{n+1}^2 u(t_1, ..., t_n, \xi_{n+1}) - \partial_{n+1} u(t_1, ..., t_n, (\mathbf{t_{i+e}})_{n+1})$$

$$+ \tfrac{1}{2} h_{n+1} (1 - v_{n+1}) \partial_{n+1}^2 u(t_1, ..., t_n, \vartheta_{n+1})$$

$$+ \sum_{\substack{\mathbf{0} \leqslant \mathbf{l} \leqslant \mathbf{e} \\ l_{n+1} = 0}} \prod_{j=1}^{n} w_j \tfrac{1}{2} h_{n+1} \partial_{n+1}^2 u(\bar{\mathbf{t}}_\mathbf{l}) \Big|$$

$$\leqslant \tfrac{1}{4} \sum_{j=1}^{n} h_j^2 \omega_d(\partial_j^2 u) + (1 - v_{n+1}) v_{n+1} h_{n+1}^2 |\partial_{n+1}^2 u(\bar{\mathbf{t}}) - \partial_{n+1}^2 u(\bar{\bar{\mathbf{t}}})|$$

$$\leqslant \tfrac{1}{4} \sum_{j=1}^{n+1} h_j^2 \omega_d(\partial_j^2 u),$$

where $\xi_{n+1}, \vartheta_{n+1} \in (t_{n+1}, t_{n+1} + h_{n+1}]$, $\bar{\mathbf{t}}_\mathbf{l}$, $\bar{\mathbf{t}}$, $\bar{\bar{\mathbf{t}}} \in [\mathbf{t_i}, \mathbf{t_{i+e}}]$, which is our first statement. For the first derivative (with respect to the first variable) we proceed similarly:

$$|\partial_1 u(\mathbf{t}) - \partial_1 S(\mathbf{t})|$$

$$= |\partial_1 u(t_1, ..., t_n, t_{n+1}) - \partial_1 S_\mathbf{i}^{(n+1)}(t_1, ..., t_n, t_{n+1})|$$

$$\leqslant v_{n+1} |\partial_1 u(t_1, ..., t_n, (\mathbf{t_{i+e}})_{n+1}) - \partial_1 S_{\mathbf{i}+\mathbf{e}_{n+1}}^{(n)}(t_1, ..., t_n)|$$

$$+ (1 - v_{n+1}) |\partial_1 u(t_1, ..., t_n, (\mathbf{t_i})_{n+1}) - \partial_1 S_\mathbf{i}^{(n)}(t_1, ..., t_n)|$$

$$+ v_{n+1}(1 - v_{n+1}) h_{n+1} |\partial_1 \partial_{n+1} u(t_1, ..., t_n, \xi_{n+1})$$

$$- \partial_1 \partial_{n+1} u(t_1, ..., t_n, \vartheta_{n+1})|$$

$$+ (1 - v_{n+1}) v_{n+1} \sum_{\substack{\mathbf{0} \leqslant \mathbf{l} \leqslant \mathbf{e} \\ l_{n+1} = 0 \\ l_1 = 0}} \prod_{j=2}^{n} w_j \frac{h_{n+1}^2}{2h_1} |\partial_{n+1}^2 u(\bar{\mathbf{t}}_\mathbf{l}) - \partial_{n+1}^2 u(\bar{\bar{\mathbf{t}}}_\mathbf{l})|$$

$$\leqslant \frac{1}{4} \sum_{j=1}^{n} h_j \omega_d(\partial_1 \partial_j u) + \frac{1}{8} \sum_{j=1}^{n} \frac{h_j^2}{h_1} \omega_d(\partial_j^2 u)$$

$$+ \frac{1}{4} h_{n+1} \omega_d(\partial_1 \partial_{n+1} u) + \frac{h_{n+1}^2}{8h_1} \omega_d(\partial_{n+1}^2 u),$$

with some appropriate $\bar{\mathbf{t}}_1$, $\bar{\bar{\mathbf{t}}}_1$. Finally, for the second derivative we get

$$\begin{aligned}
&|\partial_1^2 u(t) - \partial_1^2 S(\mathbf{t})| \\
&\quad = |\partial_1^2 u(t_1, ..., t_n, t_{n+1}) - \partial_1^2 S_{\mathbf{i}}^{(n+1)}(t_1, ..., t_n, t_{n+1})| \\
&\quad \leqslant v_{n+1} |\partial_1^2 u(t_1, ..., t_n, (\mathbf{t}_{\mathbf{i}+\mathbf{e}})_{n+1}) - \partial_1^2 S_{\mathbf{i}+\mathbf{e}_{n+1}}^{(n)}(t_1, ..., t_n)| \\
&\qquad + (1 - v_{n+1}) |\partial_1^2 u(t_1, ..., t_n, (\mathbf{t}_{\mathbf{i}})_{n+1}) - \partial_1^2 S_{\mathbf{i}}^{(n)}(t_1, ..., t_n)| \\
&\qquad + v_{n+1} |\partial_1^2 u(t_1, ..., t_n, t_{n+1}) - \partial_1^2 u(t_1, ..., t_n, (\mathbf{t}_{\mathbf{i}+\mathbf{e}})_{n+1})| \\
&\qquad + (1 - v_{n+1}) |\partial_1^2 u(t_1, ..., t_n, t_{n+1}) - \partial_1^2 u(t_1, ..., t_n, (\mathbf{t}_{\mathbf{i}})_{n+1})| \\
&\quad \leqslant n\omega_d(\partial_1^2 u) + \omega_d(\partial_1^2 u) \leqslant (n+1)\, \omega_d(\partial_1^2 u),
\end{aligned}$$

and our theorem is proved. ∎

In the applications it occurs frequently that the values of the function, which is to be approximated, are known at the knots of some subdivision, but not the values of its derivatives. In these cases the choice

$$u_{\mathbf{i}}^{(\mathbf{e}_j)} = (u_{\mathbf{i}+\mathbf{e}_j} - u_{\mathbf{i}})/h_j$$

(forward-difference) seems to be plausible. It turns out that in this case the respective spline function is piecewise linear; actually it is the generalization of the one-dimensional approximation with line segments.

Another possible definition of $u_{\mathbf{i}}^{(\mathbf{e}_j)}$ with the help of differences of function values is the following (central-difference): for all $\mathbf{i}$ in $\mathbb{Z}^n$ and for $j = 1, 2, ..., n$ we let

$$u_{\mathbf{i}}^{(\mathbf{e}_j)} = (u_{\mathbf{i}+\mathbf{e}_j} - u_{\mathbf{i}-\mathbf{e}_j})/2h_j. \tag{6$_n$}$$

Remark 2. We remark that, for instance, the coefficients of the two-dimensional quadratic spline function of the form $(1)_2$–$(4)_2$ and $(6)_2$ can be expressed as follows: for any $x \in [x_i, x_{i+1}]$ and $y \in [y_j, y_{j+1}]$ we have

$$A_{i,j}^{(0,0)} = u_{i,j}$$

$$A_{i,j}^{(1,0)} = \frac{1}{2h}(u_{i+1,j} - u_{i-1,j}), \qquad A_{i,j}^{(0,1)} = \frac{1}{2l}(u_{i,j+1} - u_{i,j-1})$$

$$A_{i,j}^{(2,0)} = \frac{1}{2h^2} \Delta^{2,0} u_{i-1,j}, \qquad A_{i,j}^{(0,2)} = \frac{1}{2l^2} \Delta^{0,2} u_{i,j-1}$$

$$A_{i,j}^{(1,1)} = \frac{1}{2hl} \{\Delta^{1,1} u_{i-1,j} + \Delta^{1,1} u_{i,j-1}\}$$

$$A_{i,j}^{(2,1)} = \frac{1}{2h^2 l} \Delta^{2,1} u_{i-1,j}, \qquad A_{i,j}^{(1,2)} = \frac{1}{2hl^2} \Delta^{1,2} u_{i,j-1},$$

where $x_{i+1} - x_i = h$, $y_{j+1} - y_j = l$.

The approximating properties of the respective spline function are expressed by the following theorems.

THEOREM 3. *Let $u\colon \mathbb{R}^n \to \mathbb{R}$ be continuous. Then the spline function $\bar{S}$ defined by the conditions $(1)_n$–$(4)_n$ and $(6)_n$ satisfies*

$$|u(\mathbf{t}) - \bar{S}(\mathbf{t})| \leqslant \tfrac{5}{4} n\omega_d(u)$$

for all $\mathbf{t}$ *in* $\mathbb{R}^n$, *where d is the diameter corresponding to the subdivision.*

Proof. In the one-dimensional case we have, for all $t \in [t_i, t_{i+1}]$,

$$|\bar{S}(t) - u(t)| = |\bar{S}_i(t) - u(t)| \leqslant |u_i - u(t)|\,(1 - v^2) + |u_{i+1} - u(t)|\, v^2 + \tfrac{1}{2}|u_{i+1} - u_{i-1}|\, v(1 - v) \leqslant \tfrac{5}{4}\,\omega_h(u),$$

where $v = (t - t_i)/h$. By induction and by the recursive formula it follows that

$$\begin{aligned}
&|\bar{S}_i^{(n+1)}(t_1, \ldots, t_n, t_{n+1}) - u(t_1, \ldots, t_n, t_{n+1})| \\
&\quad\leqslant v_{n+1}\,|u(t_1, \ldots, t_n, (\mathbf{t}_{\mathbf{i}+\mathbf{e}})_{n+1}) - \bar{S}^{(n)}_{\mathbf{i}+\mathbf{e}_{n+1}}(t_1, \ldots, t_n)| \\
&\qquad + (1 - v_{n+1})\,|u(t_1, \ldots, t_n, (\mathbf{t}_{\mathbf{i}})_{n+1}) - \bar{S}^{(n)}_{\mathbf{i}}(t_1, \ldots, t_n)| \\
&\qquad + |v_{n+1}[u(t_1, \ldots, t_n, t_{n+1}) - u(t_1, \ldots, t_n, (\mathbf{t}_{\mathbf{i}+\mathbf{e}})_{n+1})] \\
&\qquad + (1 - v_{n+1})[u(t_1, \ldots, t_n, t_{n+1}) - u(t_1, \ldots, t_n, (\mathbf{t}_{\mathbf{i}})_{n+1})]| \\
&\qquad + \tfrac{1}{2}\,(1 - v_{n+1})\, v_{n+1} \sum_{\substack{\mathbf{0} \leqslant \mathbf{l} \leqslant \mathbf{e} \\ l_{n+1} = 0}} \prod_{j=1}^{n} w_j\, |\Delta^{(2\mathbf{e}_{n+1})} u_{\mathbf{i}+\mathbf{1}-\mathbf{e}_{n+1}}| \\
&\quad\leqslant \tfrac{5}{4} n\omega_h(u) + \omega_h(u) + \tfrac{1}{4}\omega_h(u) \leqslant \tfrac{5}{4}(n+1)\,\omega_h(u). \quad ∎
\end{aligned}$$

THEOREM 4. *Let $u\colon \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable. Then the spline function $\bar{S}$ defined by the conditions $(1)_n$–$(4)_n$ and $(6)_n$ satisfies*

$$|u(\mathbf{t}) - \bar{S}(\mathbf{t})| \leqslant \tfrac{3}{4} \sum_{j=1}^{n} h_j \omega_d(\partial_j u)$$

and

$$|\partial_k u(\mathbf{t}) - \partial_k \bar{S}(\mathbf{t})| \leqslant n\omega_d(\partial_k u) + \frac{1}{h_k} \sum_{j=1}^{n} h_j \omega_d(\partial_j u)$$

for all $\mathbf{t}$ *in* $\mathbb{R}^n$ *and for $k = 1, \ldots, n$, where d is the diameter corresponding to the subdivision.*

Proof. Let S denote the spline function defined by $(1)_n$–$(5)_n$. By induction on the dimension we show that

$$|S(\mathbf{t}) - \bar{S}(\mathbf{t})| \leqslant \tfrac{1}{4} \sum_{j=1}^{n} h_j \omega_d(\partial_j u)$$

and

$$|\partial_k S(\mathbf{t}) - \partial_k \bar{S}(\mathbf{t})| \leqslant \frac{1}{2} \omega_d(\partial_k u) + \frac{1}{2h_k} \sum_{j=1}^{n} h_j \omega_d(\partial_j u)$$

holds for all $\mathbf{t}$ in $\mathbb{R}^n$ and for $k = 1, ..., n$. In the one-dimensional case we obtain, for $t \in [t_i, t_{i+1}]$,

$$|S(t) - \bar{S}(t)| = hv(1-v) \left| u_i' - \frac{u_{i+1} - u_{i-1}}{2h} \right| \leqslant \frac{1}{4} h\omega_d(u'),$$

$$|S'(t) - \bar{S}'(t)| = |1 - 2v| \left| u_i' - \frac{u_{i+1} - u_{i-1}}{2h} \right| \leqslant \omega_d(u'),$$

where $v = (t - t_i)/h$. Then it follows for all $\mathbf{t}$ in $[\mathbf{t_i}, \mathbf{t_{i+e}}]$ that

$$\begin{aligned} |S(\mathbf{t}) - \bar{S}(\mathbf{t})| &\leqslant \frac{1}{4} \sum_{j=1}^{n} h_j \omega_d(\partial_j u) \\ &\quad + (1 - v_{n+1})\, v_{n+1} h_{n+1} \sum_{\substack{\mathbf{0} \leqslant \mathbf{l} \leqslant \mathbf{e} \\ l_{n+1} = 0}} \prod_{j=1}^{n} w_j \\ &\quad \times \left| u_{\mathbf{i+1}}^{(\mathbf{e}_{n+1})} - \frac{u_{\mathbf{i+1+e}_{n+1}} - u_{\mathbf{i+1-e}_{n+1}}}{2h_{n+1}} \right| \\ &\leqslant \frac{1}{4} \sum_{j=1}^{n+1} h_j \omega_d(\partial_j u). \end{aligned}$$

Without loss of generality we prove the respective statement for the derivatives in the case $k = 1$:

$$\begin{aligned} |\partial_1 S(\mathbf{t}) - \partial_1 \bar{S}(\mathbf{t})| &\leqslant \frac{1}{2} \omega_d(\partial_1 u) + \frac{1}{2h_1} \sum_{j=1}^{n} h_j \omega_d(\partial_j u) \\ &\quad + (1 - v_{n+1})\, v_{n+1} \frac{h_{n+1}}{h_1} \sum_{\substack{\mathbf{0} \leqslant \mathbf{l} \leqslant \mathbf{e} \\ l_{n+1} = 0 \\ l_1 = 0}} \prod_{j=2}^{n} w_j 2\omega_d(\partial_{n+1} u) \\ &\leqslant \frac{1}{2} \omega_d(\partial_1 u) + \frac{1}{2h_1} \sum_{j=1}^{n+1} h_j \omega_d(\partial_j u). \end{aligned}$$

Hence, by Theorem 1,

$$|u(\mathbf{t})-\bar{S}(\mathbf{t})| \leqslant |u(\mathbf{t})-S(\mathbf{t})|+|S(\mathbf{t})-\bar{S}(\mathbf{t})| \leqslant \tfrac{3}{4} \sum_{j=1}^{n} h_j \omega_d(\partial_j u),$$

and similarly

$$\begin{aligned}|\partial_k S(\mathbf{t})-\partial_k \bar{S}(\mathbf{t})| &\leqslant |\partial_k u(\mathbf{t})-\partial_k S(\mathbf{t})|+|\partial_k S(\mathbf{t})-\partial_k \bar{S}(\mathbf{t})| \\ &\leqslant n\omega_d(\partial_k u)+\frac{1}{h_k} \sum_{j=1}^{n} h_j \omega_d(\partial_j u). \quad \blacksquare\end{aligned}$$

THEOREM 5. *Let $u\colon \mathbb{R}^n \to \mathbb{R}$ be twice continuously differentiable. Then the spline function $\bar{S}$ defined by the conditions $(1)_n$–$(4)_n$ and $(6)_n$ satisfies*

$$\begin{aligned}|u(\mathbf{t})-\bar{S}(\mathbf{t})| &\leqslant \frac{3}{8} \sum_{j=1}^{n} h_j^2 \omega_d(\partial_j^2 u), \\ |\partial_k u(\mathbf{t})-\partial_k \bar{S}(\mathbf{t})| &\leqslant \frac{1}{4} h_k \omega_d(\partial_k^2 u)+\frac{1}{4} \sum_{j=1}^{n} h_j \omega_d(\partial_k \partial_j u) \\ &\quad +\frac{3}{8h_k} \sum_{j=1}^{n} h_j^2 \omega_d(\partial_j^2 u), \\ |\partial_k^2 u(\mathbf{t})-\partial_k^2 \bar{S}(\mathbf{t})| &\leqslant (n+1)\, \omega_d(\partial_k^2 u)\end{aligned}$$

for all $\mathbf{t}$ *in* $\mathbb{R}^n$ *and for* $k=1, ..., n$, *where* d *is the diameter corresponding to the subdivision.*

Proof. Let S denote the spline function defined by $(1)_n$–$(5)_n$. By induction on the dimension we show that

$$|S(\mathbf{t})-\bar{S}(\mathbf{t})| \leqslant \frac{1}{8} \sum_{j=1}^{n} h_j^2 \omega_d(\partial_j^2 u)$$

and

$$\begin{aligned}|\partial_k S(\mathbf{t})-\partial_k \bar{S}(\mathbf{t})| &\leqslant \frac{1}{4} h_k \omega_d(\partial_k^2 u)+\frac{1}{4h_k} \sum_{j=1}^{n} h_j^2 \omega_d(\partial_j^2 u), \\ |\partial_k^2 S(\mathbf{t})-\partial_k^2 \bar{S}(\mathbf{t})| &\leqslant \omega_d(\partial_k^2 u)\end{aligned}$$

holds for all $\mathbf{t}$ in $\mathbb{R}^n$ and for $k=1, ..., n$. In the one-dimensional case we have, for $t \in [t_i, t_{i+1}]$,

$$|S(t)-\bar{S}(t)| = hv(1-v)\left|u_i' - \frac{u_{i+1}-u_{i-1}}{2h}\right| \leqslant \frac{1}{8}h^2\omega_d(u''),$$

$$|S'(t)-\bar{S}'(t)| = |1-2v|\left|u_i' - \frac{u_{i+1}-u_{i-1}}{2h}\right| \leqslant \frac{1}{2}h\omega_d(u''),$$

$$|S''(t)-\bar{S}''(t)| = \frac{2}{h}\left|u_i' - \frac{u_{i+1}-u_{i-1}}{2h}\right| \leqslant \omega_d(u''),$$

where $v=(t-t_i)/h$. In the $n+1$-dimensional case, similarly to the previous theorem, we obtain, for all $\mathbf{t}$ in $[\mathbf{t}_\mathbf{i}, \mathbf{t}_{\mathbf{i}+\mathbf{e}}]$,

$$\begin{aligned}|S(\mathbf{t})-\bar{S}(\mathbf{t})| &\leqslant \frac{1}{8}\sum_{j=1}^{n} h_j^2\omega_d(\partial_j^2 u)\\ &\quad + (1-v_{n+1})\,v_{n+1}h_{n+1}\sum_{\substack{\mathbf{0}\leqslant\mathbf{l}\leqslant\mathbf{e}\\ l_{n+1}=0}}\prod_{j=1}^{n} w_j\\ &\quad \times\left|u_{\mathbf{i}+\mathbf{1}}^{(\mathbf{e}_{n+1})} - \frac{u_{\mathbf{i}+\mathbf{1}+\mathbf{e}_{n+1}} - u_{\mathbf{i}+\mathbf{1}-\mathbf{e}_{n+1}}}{2h_{n+1}}\right|\\ &\leqslant \frac{1}{8}\sum_{j=1}^{n+1} h_j^2\omega_d(\partial_j^2 u)\end{aligned}$$

and

$$\begin{aligned}|\partial_1 S(\mathbf{t})-\partial_1\bar{S}(\mathbf{t})| &\leqslant \frac{1}{4}h_1\omega_d(\partial_1^2 u) + \frac{1}{4h_1}\sum_{j=1}^{n} h_j^2\omega_d(\partial_j^2 u)\\ &\quad + (1-v_{n+1})\,v_{n+1}\frac{h_{n+1}^2}{h_1}\sum_{\substack{\mathbf{0}\leqslant\mathbf{l}\leqslant\mathbf{e}\\ l_{n+1}=0\\ l_1=0}}\prod_{j=2}^{n} w_j\omega_d(\partial_{n+1}^2 u)\\ &\leqslant \frac{1}{4}h_1\omega_d(\partial_1^2 u) + \frac{1}{4h_1}\sum_{j=1}^{n+1} h_j^2\omega_d(\partial_j^2 u).\end{aligned}$$

Hence, by the estimations in Theorem 2 we get our statement. ∎

For practical reasons it is useful to prove the stability of our construction, which means that small perturbations in the initial data have only a small effect on the resulting spline function.

THEOREM 6. *Let S, resp. $\bar{S}$, denote the spline function defined by*

$(1)_n$–$(4)_n$ and $(6)_n$ corresponding to the systems $\{u_{\mathbf{i}}\}$ and $\{\bar{u}_{\mathbf{i}}\}$, respectively, where $|u_{\mathbf{i}} - \bar{u}_{\mathbf{i}}| \leqslant \varepsilon$ holds for all $\mathbf{i}$. *Then we have*

$$|S(\mathbf{t}) - \bar{S}(\mathbf{t})| \leqslant \left(n + \frac{3}{2}\right) \frac{\varepsilon}{2}$$

for all $\mathbf{t}$ *in* $\mathbb{R}^n$.

Proof. In the one-dimensional case we obviously have

$$|S(t) - \bar{S}(t)| \leqslant \varepsilon(1 - v^2) + \varepsilon v^2 + v(1 - v)\varepsilon \leqslant \tfrac{5}{4}\varepsilon;$$

hence our statement is an immediate consequence of the recursive formula (see Remark 1). ∎

Remark 3. The question of how to define the spline function at the edge of a bounded domain, that is, in the case where the function values for the differences are not defined, arises. Let, for instance, the bounded region be $[\mathbf{a}, \mathbf{b}]$ and we divide the interval $[a_j, b_j]$ into m_j equal pieces, $h_j = (b_j - a_j)/m_j$; that is, the jth coordinates of the knots are $a_j = t_{j0} < t_{j1} < \cdots < t_{jm_j} = b_j$. Let

$$S_{\mathbf{i}}(\mathbf{t}) = S_{\mathbf{i} + \mathbf{e}_j}(\mathbf{t}) \qquad \text{for} \quad (\mathbf{i})_j = 0.$$

For instance, in the two-dimensional case we have the following for all i, j,

$$S_{0,j}(t, s) = S_{1,j}(t, s)$$
$$S_{i,0}(t, s) = S_{i,1}(t, s)$$
$$S_{0,0}(t, s) = S_{1,1}(t, s).$$

As our estimates are based on the Taylor formula, here the order of the approximation is the same. Further, this spline function is continuous on $[\mathbf{a}, \mathbf{b}]$.

4. Shape-Preserving Properties

Recently special attention has been paid to shape preserving properties of approximation methods in one and several dimensions (see, e.g., [2, 3, 7, 8]). The following show that our construction has some shape-preserving properties, too.

THEOREM 7. *Suppose that the given system* $\{u_i\}$ *satisfies*

$$\Delta^{(2\mathbf{e}_j)} u_{\mathbf{i}} \geqslant 0$$

for all $\mathbf{i}$ *and for some j. Then the spline function* $\bar{S}$ *defined by the conditions* $(1)_n$–$(4)_n$ *and* $(6)_n$ *is convex on each rectangle of the subdivision in the jth variable.*

Proof. By the recursive formula one gets

$$\begin{aligned}\bar{S}_{\mathbf{i}}^{(n+1)}&(t_1, ..., t_n, t_{n+1})\\ &= v_{n+1} \bar{S}_{\mathbf{i}+\mathbf{e}_{n+1}}^{(n)}(t_1, ..., t_n) + (1 - v_{n+1}) \bar{S}_{\mathbf{i}}^{(n)}(t_1, ..., t_n)\\ &+ \tfrac{1}{2}(v_{n+1} - 1) v_{n+1} \sum_{\substack{\mathbf{0} \leqslant \mathbf{l} \leqslant \mathbf{e}\\ l_{n+1}=0}} \sum_{j=1}^{n} w_j \Delta^{(2\mathbf{e}_{n+1})} u_{\mathbf{i}+\mathbf{l}-\mathbf{e}_{n+1}},\end{aligned}$$

where

$$v_j = \frac{(\mathbf{t})_j - (\mathbf{t}_{\mathbf{i}})_j}{h_j}, \qquad w_j = \begin{cases} v_j, & \text{if } l_j = 1\\ 1 - v_j, & \text{if } l_j = 0\end{cases}$$

for $j = 1, ..., n+1$. By assumption it follows that

$$\begin{aligned}\partial_{n+1}^2 \bar{S}_{\mathbf{i}}^{(n+1)}&(t_1, ..., t_n, t_{n+1})\\ &= \frac{1}{h_{n+1}^2} \sum_{\substack{\mathbf{0} \leqslant \mathbf{l} \leqslant \mathbf{e}\\ l_{n+1}=0}} \prod_{j=1}^{n} w_j \Delta^{(2\mathbf{e}_{n+1})} u_{\mathbf{i}+\mathbf{l}-\mathbf{e}_{n+1}} \geqslant 0,\end{aligned}$$

which is the statement. ∎

The above statement can be strengthened in the two-dimensional case as follows.

THEOREM 8. *Suppose that the given system* $\{u_{i,j}\}$ *satisfies*

$$\Delta^{\alpha,\beta} u_{i,j} \geqslant 0 \qquad (0 \leqslant \alpha, \beta \leqslant 2, \alpha + \beta \leqslant 2)$$

for all i, j. Then the spline function $\bar{S}$ *defined by the conditions* $(1)_2$–$(4)_2$ *and* $(6)_2$ *is nonnegative, monotonically increasing in each variable. Moreover, if in addition*

$$\Delta^{1,1} u_{i,j} \geqslant \Delta^{1,1} u_{i,j-1}, \qquad \Delta^{1,1} u_{i,j} \geqslant \Delta^{1,1} u_{i-1,j},$$

then $\partial_1 \partial_2 S(x, y) \geqslant 0$ *on each rectangle.*

Proof. Let $x_i \leqslant x \leqslant x_{i+1}$ and $y_j \leqslant y \leqslant y_{j+1}$; then we have

$$\partial_1^2 S_{i,j}(x, y) = \frac{2}{2h^2} \Delta^{2,0} u_{i-1,j}(1-s) + \frac{2}{2h^2} \Delta^{2,0} u_{i-1,j+1} s \geqslant 0,$$

where $x_{i+1}-x_i=h$, $y_{j+1}-y_j=l$, $t=(x-x_i)/h$, and $s=(y-y_j)/l$. As $\partial_1 S_{i,j}(x, y)$ is monotonically increasing in the variable x, to prove its nonnegativity, it is enough to show that the function

$$p(y)=\partial_1 S_{i,j}(x_i, y)=\frac{1}{2h}(u_{i+1,j}-u_{i-1,j})$$

$$+\frac{1}{2hl}\{\Delta^{1,1}u_{i-1,j}+\Delta^{1,1}u_{i,j-1}\}(y-y_j)$$

$$+\frac{1}{2hl^2}\Delta^{1,2}u_{i,j-1}(y-y_j)^2$$

is nonnegative on $[y_j, y_{j+1}]$. The quadratic polynomial p is nonnegative at y_j and y_{j+1}. If $\Delta^{1,2}u_{i,j-1}>0$, then p is convex and its minimum is outside of the interval $[y_j, y_{j+1}]$; hence it is monotonic on it. If $\Delta^{1,2}u_{i,j-1}\leqslant 0$, then p is concave on $[y_j, y_{j+1}]$; thus our statement follows in both cases. The nonnegativity of the function S is a consequence of the interpolation property and the monotonicity. Finally, if $\Delta^{1,1}u_{i,j}\geqslant\Delta^{1,1}u_{i,j-1}$ and $\Delta^{1,1}u_{i,j}\geqslant\Delta^{1,1}u_{i-1,j}$, then

$$\partial_1\partial_2 S_{i,j}(x, y)=\frac{1}{2hl}\{\Delta^{1,1}u_{i-1,j}+\Delta^{1,1}u_{i,j-1}\}+\frac{1}{hl}\Delta^{2,1}u_{i-1,j}t$$

$$+\frac{1}{hl}\Delta^{1,2}u_{i,j-1}v\geqslant 0. \quad \blacksquare$$

5. Applications

A well-known and useful numerical method for the approximate solution of partial differential equations is the method of finite differences (net method). In this method the partial derivatives of the unknown function in the equation are replaced by appropriate differences on a given rectangular grid. Hence the original problem converts into a difference equation for the approximate values of the unknown function at the knots. A possible discretization in a second order partial differential equation is

$$\partial_j u(\mathbf{t_i})\approx u_{\mathbf{i}}^{(\mathbf{e}_j)}=(u_{\mathbf{i}+\mathbf{e}_j}-u_{\mathbf{i}-\mathbf{e}_j})/2h_j,$$

$$\partial_j^2 u(\mathbf{t_i})\approx(u_{\mathbf{i}+\mathbf{e}_j}-2u_{\mathbf{i}}+u_{\mathbf{i}-\mathbf{e}_j})/h_j^2,$$

$$\partial_j\partial_k u(\mathbf{t_i})\approx(u_{\mathbf{i}+\mathbf{e}_j+\mathbf{e}_k}-u_{\mathbf{i}+\mathbf{e}_j}-u_{\mathbf{i}+\mathbf{e}_k}+u_{\mathbf{i}})/2h_jh_k.$$

(see, e.g., [1, 10]). In this case the quadratic spline function defined by $(1)_n$–$(4)_n$ and $(6)_n$ is a solution of the original problem at the knots, as its partial derivatives at the knots are given exactly by the above formulas. By the stability property our spline function can be considered an approximate solution on the whole domain, and if the error at the knots is at most ε, then by Theorems 5 and 6 the error on the whole domain is at most $\frac{1}{2}\sum_{j=1}^{n} h_j^2\omega_d(\partial_j^2 u) + (n+\frac{3}{2})(\varepsilon/2)$. This approximation procedure can be applied, for instance, for elliptic, parabolic, and hyperbolic partial differential equations, even in the nonlinear case.

6. Numerical Examples

Here we give some numerical examples which may illustrate the possible applications of the above methods. In the first example we present a graphical comparison of the exact and spline approximations defined by the conditions $(1)_2$–$(4)_2$ and $(6)_2$ for the functions

$$u(x, y) = [1 + 2e^{-3(r-6.7)}]^{-1/2}, \qquad r = (x^2 + y^2)^{1/2}$$

on $[0, 10]^2$ (Fig. 1) and

$$u(x, y) = \begin{cases} |x|\, y, & xy \geqslant 0, \\ 0, & \text{otherwise} \end{cases}$$

u(x,y)=[1+2e^{-3(r-6.7)}]^{-1/2} , r=(x^2+y^2)^{1/2} , x,y ∈ [0,10]

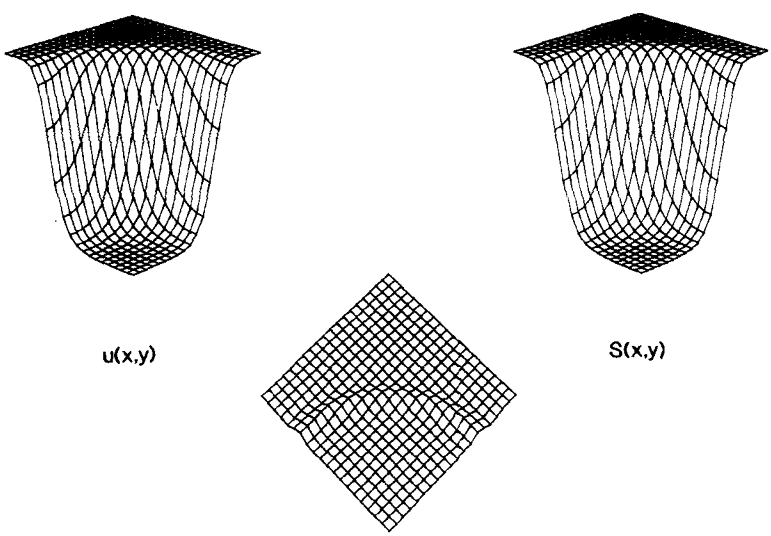


Figure 1

on $[-4, 4]^2$ (Fig. 2). The intervals have been divided into 20 subintervals in both cases. We note that we have plotted the error functions $u-S$ in both cases, too (see also [2]).

The second example is the approximation of the function

$$u_0(x, y, z) = xye^{x+z}$$

by the spline function of the form $(1)_3$–$(4)_3$ and $(6)_3$ on the unit cube $[0, 1]^3$. We have used a subdivision corresponding to $h_1 = h_2 = h_3 = 10^{-2}$. By Theorem 2 the error of the approximation is

$$|u_0(x, y, z) - S(x, y, z)| \leqslant 2.58617\ 10^{-5} \qquad \text{on } [0, 1]^3.$$

Some exact and approximate values are summarized in Table 1.

In Table 2 the approximation of Δu_0 by ΔS on the above grid is illustrated (Δ denotes the Laplace operator).

The third example is the finite-difference approximation to the hyperbolic "wave" equation (see [11, p. 199])

$$\partial_1^2 u(x, t) = \partial_2^2 u(x, t)$$

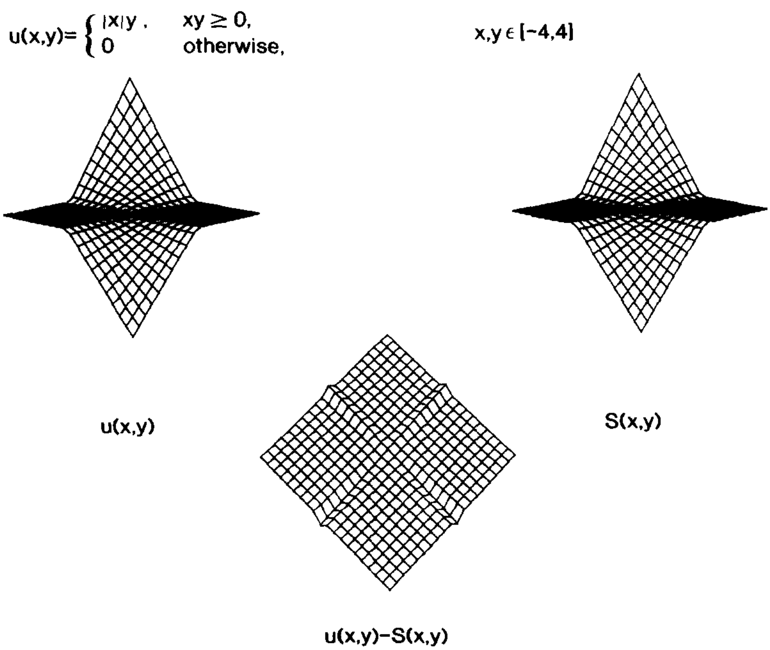


FIGURE 2

TABLE 1

x	y	z	$u_0(x, y, z)$	$S(x, y, z)$
0.362	0.477	0.409	3.7330860011$E-01$	3.7330872132$E-01$
0.362	0.477	0.818	5.6194581111$E-01$	5.6194600437$E-01$
0.362	0.954	0.409	7.4661720022$E-01$	7.4661744265$E-01$
0.362	0.954	0.818	1.1238916222$E+00$	1.1238920087$E+00$
0.724	0.477	0.409	1.0722908330$E+00$	1.0722911708$E+00$
0.724	0.477	0.818	1.6141319587$E+00$	1.6141324983$E+00$
0.724	0.954	0.409	2.1445816660$E+00$	2.1445823417$E+00$
0.724	0.954	0.818	3.2282639174$E+00$	3.2282649966$E+00$

with the boundary conditions

$$u(0, t)=0 \quad \text{and} \quad u(1, t)=0 \quad (t \geqslant 0)$$

and the initial conditions

$$u(x, 0)=\tfrac{1}{8}\sin \pi x, \quad \partial_2 u(x, 0)=0 \quad (0 \leqslant x \leqslant 1).$$

The analytical solution is

$$u(x, t)=\tfrac{1}{8}\sin \pi x \cos \pi t \quad (0 \leqslant x \leqslant 1,\ t \geqslant 0).$$

The explicit finite-difference formula of the equation with steps $h=l$ and the central-difference approximation for the derivative condition give the following recursive formulas in the grid points:

$$u_{i,j+1}=u_{i-1,j}+u_{i+1,j}-u_{i,j-1} \quad (j \geqslant 1),$$
$$u_{i,1}=(u_{i-1,0}+u_{i+1,0})/2.$$

Let $h=l=0.1$, and using these approximative function values at the grid

TABLE 2

x	y	z	$\Delta u_0(x, y, z)$	$\Delta S(x, y, z)$
0.362	0.477	0.409	2.8090956539$E+00$	2.7988966219$E+00$
0.362	0.477	0.818	4.2285646118$E+00$	4.2137818940$E+00$
0.362	0.954	0.409	5.6181913078$E+00$	5.5977932314$E+00$
0.362	0.954	0.818	8.4571292235$E+00$	8.4275637906$E+00$
0.724	0.477	0.409	5.1067110390$E+00$	5.0752305402$E+00$
0.724	0.477	0.818	7.6871919801$E+00$	7.6414257201$E+00$
0.724	0.954	0.409	1.0213422078$E+01$	1.0150461004$E+01$
0.724	0.954	0.818	1.5374383960$E+01$	1.5282851443$E+01$

TABLE 3

x	t	$u(x, t)$	$\bar{S}(x, t)$	$u(x, t) - \bar{S}(x, t)$
0.5	0.5	0.0625	0.0625	−0.0000
0.9	0.5	0.0873	0.0874	0.0001
1.3	0.5	0.0788	0.0789	0.0002
1.7	0.5	0.0401	0.0403	0.0002
0.5	0.9	0.0138	0.0140	0.0001
0.9	0.9	0.0193	0.0195	0.0002
1.3	0.9	0.0174	0.0176	0.0002
1.7	0.9	0.0089	0.0090	0.0001
0.5	1.3	−0.0401	−0.0399	0.0002
0.9	1.3	−0.0561	−0.0558	0.0003
1.3	1.3	−0.0506	−0.0504	0.0002
1.7	1.3	−0.0258	−0.0257	0.0000
0.5	1.7	−0.0788	−0.0785	0.0003
0.9	1.7	−0.1100	−0.1098	0.0002
1.3	1.7	−0.0992	−0.0992	0.0001
1.7	1.7	−0.0506	−0.0507	−0.0001

points we construct the spline function $\bar{S}$ defined by the conditions $(1)_n$–$(4)_n$ and $(6)_n$. Table 3 contains the values of the exact and numerical solutions by the spline function $\bar{S}$ rounding to $4D$ on $[0, 1]^2$.

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